UNRAMIFIED REDUCTORS OF FILTERED AND GRADED ALGEBRAS

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Abstract. It is a fairly known fact that most of the algebras appearing in the theory of rings of differential operators, quantized algebras of different kinds (including many quantum groups), regular algebras in projective non-commutative geometry, etc... come equipped with a natural gradation or filtration controlled by some finite dimensional vector space(s), e.g. the degree one part of filtration or gradation. In this note we relate the valuations of the algebras considered to unramified sub-lattices in some vector space(s).

Introduction

In the attempt to finding "good" reductors of filtered and/or graded algebras we found out that the ones which satisfy an *unramifiedness* condition are in fact the most suitable for our purposes, among them the most important fact being that they give rise to a separated filtration that comes form the given valuation on the ground field. Starting form this observation we introduce the notion of the unramified F-reductor, respectively unramified graded reductor and reduce the problem of finding an extending non-commutative valuation to finding a reductor in an associated graded ring having a domain for its reduction. As an application we single out the case of affine algebras, respectively positively graded algebras although our results apply to a large class of examples.

Throughout this paper we consider K as being a field, Γ a totally ordered group and $v : K \longrightarrow \Gamma \cup \{\infty\}$ a valuation on K. We always assume that v is surjective, hence Γ is a commutative group. Set O_v the valuation ring of K associated to v, m_v its unique maximal ideal and $k_v = O_v/m_v$ the residue field. For a given ring R a family FR of additive subgroups $F_\gamma R$, $\gamma \in \Gamma$ satisfying

(*i*) $\gamma \leq \delta$ implies $F_\gamma R \subseteq F_\delta R$,

(*ii*) $F_\gamma R\, F_\delta R \subseteq F_{\gamma+\delta} R$, for all $\gamma, \delta \in \Gamma$,

(*iii*) $1 \in F_0 R$,

is a called a Γ-*filtration* on R. For a Γ-filtration FR we may define the associated graded ring $G_F(R) = \oplus_{\gamma\in\Gamma} F_\gamma R/F_{<\gamma}R$, where $F_{<\gamma}R = \sum_{\gamma'<\gamma} F_{\gamma'}R$. We say that FR is Γ-*separated* if for every $a \in R$, $a \neq 0$ there is a $\gamma \in \Gamma$ such that $a \in F_\gamma R - F_{<\gamma}R$. Note that for $\Gamma = \mathbb{Z}$ the Γ-separatedness is equivalent to the separatedness, that is $\cap_{n\in\mathbb{Z}} F_n R = 0$, but for an arbitrary Γ the latter condition may be strictly weaker. If

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FR is Γ-separated, then we may define the *principal symbol map* $\sigma : R \longrightarrow G_F(R)$ by $\sigma(a) = a \bmod F_{<\gamma}R$ whenever $a \in F_\gamma R - F_{<\gamma}R$. The *degree* $\deg \sigma(a) = \gamma$ of $\sigma(a)$ is uniquely determined. To FR one associates a value-function $v_F : R \longrightarrow \Gamma \cup \{\infty\}$ defined by $v_F(0) = \infty$ and $v_F(a) = -\deg(a)$ for $a \neq 0$. It is well known that v_F is a valuation function on R whenever $G_F(R)$ is a domain; see [7, Corollary 4.2.7]. On the other side, for a valuation v on K as before we consider a Γ-filtration f^vK on K given by $f^v_\gamma K = \{x \in K : v(x) \geq -\gamma\}$. Obviously $f^v_0 K = O_v$ and $f^v_{<0}K = m_v$. All the filtrations considered in this paper are supposed to be *exhaustive*, that is $\cup_{\gamma\in\Gamma} F_\gamma R = R$. For other unexplained notations the reader is referred to [7].

1. Unramified reductions of filtered and graded algebras

Definition 1.1. *If Δ is a skewfield and $\Lambda \subset \Delta$ is a subring, then Λ is a* valuation ring *of Δ if it is invariant under inner automorphisms of Δ and for $x \in \Delta$, $x \neq 0$, either $x \in \Lambda$ or $x^{-1} \in \Lambda$.*

Let V be a finite dimensional K-vector space and M an O_v-submodule of V. Recall that M is an *O_v-lattice* in V if it contains a K-basis of V and is a submodule of a finitely generated O_v-submodule of V. Usually, we will denote the k_v-vector space M/m_vM by $\overline{V}$ and the residue field k_v by $\overline{K}$. For any O_v-lattice M in V we have $\dim_{\overline{K}} \overline{V} \leq \dim_K V$. When equality holds we say that M defines an *unramified reduction* of V.

Proposition 1.2. *Let V be a finite dimensional K-vector space and M an O_v-lattice in V.*

(i) If M is a finitely generated O_v-module, then M is a free O_v-module of rank less or equal than $\dim V$*. (This happens, for instance, when $\Gamma = \mathbb{Z}$.)*
(ii) If M defines an unramified reduction of V, then M is a free O_v-module of rank equal to $\dim V$*.*

Proof. (i) It is a well-known fact that finitely generated torsion-free modules over valuation rings are free. Since the O_v-module M is contained in a K-vector space, it is torsion-free.
(ii) Choose a $\overline{K}$-basis $\{\overline{x}_1, \ldots, \overline{x}_n\}$ in $\overline{V}$, $n = \dim_K V$. The elements $x_1, \ldots, x_n \in M$ are linearly independent over K, in particular over O_v. Now we show that $x_1, \ldots, x_n$ is a system of generators for M as an O_v-module. (Note that we can not apply Nakayama's Lemma because we do not know that M is a finitely generated O_v-module.) For an element $x \in M$ there exist $a_1, \ldots, a_n \in K$ such that $x = \sum_{i=1}^n a_i x_i$. There also exists an element $a_j \neq 0$ such that $a_i^* = a_j^{-1} a_i \in O_v$ for all i. If $a_j \in O_v$, then all the elements a_i belong to O_v, henceforth $x \in O_v x_1 + \cdots + O_v x_n$. Otherwise, $a_j^{-1} \in m_v$ and by taking the residues modulo m_vM we get $0 = \sum_{i=1}^n \widehat{a}_i^* \overline{x}_i$, a contradiction since we have $a_j^* = 1$. □

The *valuation filtration* F^vV on V defined by $F^v_\gamma V = (f^v_\gamma K)M$, $\gamma \in \Gamma$, is an exhaustive filtration, but *we do not know if it is Γ-separated or not.* As we see bellow the valuation filtration is Γ-separated whenever M is a free O_v-module.

Lemma 1.3. *If M is a free O_v-module, then the valuation filtration F^vV on V is Γ-separated.*

Proof. Let $\{x_1, \ldots, x_r\}$ be an O_v-basis of M. Then it is easily seen that $\{x_1, \ldots, x_r\}$ are linearly independent over K. Take an element $x \in V$. From $KM = V$ it follows that there exists an $a \in O_v - \{0\}$ such that $ax \in M$. Write $ax = \sum_{i=1}^r a_i x_i$ with $a_i \in O_v$ for all i. If we consider $a_i^* = a^{-1} a_i \in K$, then we can write $x = \sum_{i=1}^r a_i^* x_i$. For those $a_i^* \neq 0$ we denote $v(a_i^*)$ by $-\gamma_i$, where $\gamma_i \in \Gamma$ for all i. Assume that $\gamma_1 \leq \cdots \leq \gamma_r = \gamma$ and show that $x \in F_\gamma^v V - F_{<\gamma}^v V$. Obviously $x \in F_\gamma^v V$, and if we suppose that $x \in F_{<\gamma}^v V$, then there exists an $\delta < \gamma$ such that $x \in (f_\delta^v K)M$. Now we can write $x = \sum_{i=1}^r b_i x_i$ with $b_i \in f_\delta^v K$ for all i. This entails that $b_i = a_i^*$ for all i, in particular $b_r = a_r^*$, and thus we get $v(b_r) = -\gamma$. As $b_r \in f_\delta^v K$ we have $-\gamma \geq -\delta$, that is $\delta \geq \gamma$, a contradiction. □

Now let us consider A a K-algebra and FA an exhaustive separated $\mathbb{Z}$-filtration on A such that $K \subseteq F_0 A$. Suppose in addition that FA is *finite*, i.e. $\dim_K F_n A < \infty$ for all $n \in \mathbb{Z}$. Since FA is finite and separated, it must be *left limited*, i.e. there is $n_0 \in \mathbb{Z}$ such that $F_n A = 0$ for all $n \leq n_0$. Without loss of generality we may suppose that the filtration FA is *positive*, that is $F_n A = 0$ for all $n < 0$. Consider $\Lambda \subset A$ a subring. The *induced filtration* $F\Lambda$ of FA on Λ is given by $F_n \Lambda = \Lambda \cap F_n A$, $n \in \mathbb{N}$. Then Λ is an *F-reductor* of A if $\Lambda \cap K = O_v$ and $F_n \Lambda$ is an O_v-lattice in $F_n A$ for all $n \in \mathbb{N}$. We call the ring $\overline{A} = \Lambda / m_v \Lambda$ the (filtered) *reduction* of A with respect to Λ. The *valuation filtration* $F^v A$ on A is defined by $F_\gamma^v A = (f_\gamma^v K)\Lambda$, $\gamma \in \Gamma$.

Definition 1.4. *Let A be a filtered K-algebra with a finite filtration FA and $\Lambda \subset A$ an F-reductor. We say that Λ is an* unramified F-reductor *if $F_n \Lambda$ is an unramified reduction of $F_n A$ for all $n \in \mathbb{N}$.*

The existence of (unramified) F-reductors in the general case seems to be unlikely, but in case the algebra is given by a finite number of generators and finitely many relations that reduce well, it is easy to find one. Note that finite dimensional algebras over fields have always unramified reductors; see [6, Proposition 1.2]. When such a reductor there exists the valuation filtration on A turns out to be Γ-separated.

Proposition 1.5. *Let A be a filtered K-algebra with a finite filtration FA and $\Lambda \subset A$ an unramified F-reductor. Then the valuation filtration $F^v A$ on A is Γ-separated.*

Proof. $F_n \Lambda$ is a free O_v-module for all $n \in \mathbb{N}$. From Lemma 1.3 we get that the valuation filtration $F^v(F_n A)$ is Γ-separated for all $n \in \mathbb{N}$, and by using [1, Proposition 3.5(ii)] we have that $F^v A$ is Γ-separated. □

Note that for $\Gamma = \mathbb{Z}$ an F-reductor necessarily defines a separated filtration. In this case $F_n \Lambda$ is a finitely generated O_v-module, hence it is free; see Proposition 1.2(i).

In the following we prove some properties of the unramified F-reductors. First we show that the unramified F-reductors are free O_v-modules.

Lemma 1.6. *Let Λ be an F-reductor of A. Then we have $m_v F_j \Lambda \cap F_i \Lambda = m_v F_i \Lambda$ for all $i, j \in \mathbb{N}$.*

Proof. The non-trivial case is $i < j$. In the proof of Proposition 3.5 from [1] we showed that

$$(f_\gamma^v K)\Lambda \cap F_n A = (f_\gamma^v K)(\Lambda \cap F_n A) \tag{1}$$

for all $\gamma \in \Gamma$ and $n \in \mathbb{N}$. From (1) we can easily deduce that $m_v(\Lambda \cap F_nA) = m_v\Lambda \cap F_nA$. So $m_vF_j\Lambda \cap F_i\Lambda = m_v(\Lambda \cap F_jA) \cap F_i\Lambda = (m_v\Lambda \cap F_jA) \cap F_i\Lambda = m_v\Lambda \cap F_iA \cap \Lambda = m_v\Lambda \cap F_iA = m_v(\Lambda \cap F_iA) = m_vF_i\Lambda$. □

Proposition 1.7. *Let A be a filtered K-algebra with a finite filtration FA and $\Lambda \subset A$ an unramified F-reductor. Then Λ is a free O_v-module.*

Proof. Let $\{\overline{x}_1, \dots, \overline{x}_{p_0}\} \subset \overline{F_0A} = F_0\Lambda/m_vF_0\Lambda$ be a $\overline{K}$-basis. Then $\{x_1, \dots, x_{p_0}\}$ is an O_v-basis for $F_0\Lambda$ and a K-basis for F_0A. Since $m_vF_1\Lambda \cap F_0\Lambda = m_vF_0\Lambda$ we get that $\overline{F_0A}$ is a $\overline{K}$-vector subspace of $\overline{F_1A}$. Now we extend the $\overline{K}$-basis $\{\overline{x}_1, \dots, \overline{x}_{p_0}\}$ of $\overline{F_0A}$ to a $\overline{K}$-basis $\{\overline{x}_1, \dots, \overline{x}_{p_0}, \dots, \overline{x}_{p_1}\}$ of $\overline{F_1A}$. Again we have that $\{x_1, \dots, x_{p_0}, \dots, x_{p_1}\}$ is an O_v-basis for $F_1\Lambda$ and a K-basis for F_1A. We can continue this way and we obtain a sequence $(x_n)_{n\geq 1}$ of elements in Λ that forms an O_v-basis of Λ. □

The next result shows that the unramifiedness is preserved by taking the intersection with a filtered subring.

Proposition 1.8. *Let A be a filtered K-algebra with a finite filtration FA, A' a K-subalgebra of A and $\Lambda \subset A$ an unramified F-reductor. Then $\Lambda' = \Lambda \cap A'$ is an unramified F-reductor of A'.*

Proof. We consider on A' the induced filtration $FA' = FA \cap A'$, and similarly for Λ' we have $F\Lambda' = \Lambda' \cap FA' = \Lambda \cap A' \cap FA = F\Lambda \cap A'$. We have to prove that $F_n\Lambda'$ is an unramified reduction of F_nA' for all $n \in \mathbb{N}$. It is enough to show that $\dim_{\overline{K}} \overline{F_nA'} = \dim_K F_nA'$. In order to do that, let us first note that $m_v(\Lambda \cap F_nA \cap A') = m_v\Lambda \cap F_nA \cap A'$, or equivalently $m_vF_n\Lambda \cap F_n\Lambda' = m_vF_n\Lambda'$. It follows that $\overline{F_nA'}$ is a $\overline{K}$-vector subspace of $\overline{F_nA}$. Now we choose a $\overline{K}$-basis $\{\overline{x}_1, \dots, \overline{x}_r\}$ in $\overline{F_nA'}$ and check that the representatives $\{x_1, \dots, x_r\}$ are also a K-basis of F_nA'. Let us first extend $\{\overline{x}_1, \dots, \overline{x}_r\}$ to $\{\overline{x}_1, \dots, \overline{x}_s\}$, $s \geq r$, a $\overline{K}$-basis in $\overline{F_nA}$. Since $F_n\Lambda$ is an unramified reduction of F_nA, we have that $\{x_1, \dots, x_s\}$ is a K-basis of F_nA. Assume that $\{x_1, \dots, x_r\}$ is not a K-basis of F_nA', and extend it to $\{x_1, \dots, x_r, y_1, \dots, y_t\}$ a K-basis of F_nA'. Now we can write $y_u = \sum_{j=1}^{r} a_{uj}x_j + \sum_{k=r+1}^{s} b_{uk}x_k$, where $1 \leq u \leq t$, $a_{uj} \in K$, and $b_{uk} \in K$ not all zero. Set $z_u = \sum_{k=r+1}^{s} b_{uk}x_k$, $1 \leq u \leq t$. Obviously, $z_u \in F_nA'$ for all $u = 1, \dots, t$, and moreover $\{x_1, \dots, x_r, z_1, \dots, z_t\}$ is a K-basis of F_nA'. Let $b \in K$, $b \neq 0$, such that $b^*_{uk} = bb_{uk} \in O_v$ for all $u = 1, \dots, t$, $k = r+1, \dots, s$, $b^*_{uk_0} = 1$ for a $k_0 \in \{r+1, \dots, s\}$, and set $z^*_u = bz_u$. The elements z^*_u belong to $F_nA' \cap F_n\Lambda = F_n\Lambda'$, and $\overline{z}^*_u = \sum_{k=r+1}^{s} \widehat{b}^*_{uk}\overline{x}_k$. On the other hand, $\overline{z}^*_u \in \overline{F_nA'}$ implies that $\overline{z}^*_u = \sum_{i=1}^{r} \widehat{c}_i\overline{x}_i$ for some $\widehat{c}_i \in \overline{K}$, a contradiction. □

The unramifiedness also behaves well with respect to the tensor products.

Proposition 1.9. *Let A, A' be filtered K-algebras with finite filtrations FA, respectively FA', and $\Lambda \subset A$, respectively $\Lambda' \subset A'$ unramified F-reductors. Then $\Lambda \otimes_{O_v} \Lambda'$ is an unramified F-reductor of $A \otimes_K A'$ with respect to the tensor filtration.*

Proof. Note that $A = K \otimes_{O_v} \Lambda$, and $A' = K \otimes_{O_v} \Lambda'$. This entails that $A \otimes_K A' = (K \otimes_{O_v} \Lambda) \otimes_K (K \otimes_{O_v} \Lambda') = K \otimes_{O_v} (\Lambda \otimes_{O_v} \Lambda')$, and thus we can deduce that $\Lambda \otimes_{O_v} \Lambda'$ is a subring of $A \otimes_K A'$. Now we define on the tensor product $A \otimes_K A'$ a finite filtration (called *tensor* filtration) given by $F_n(A \otimes_K A') = \bigoplus_{i+j=n} F_iA \otimes_K F_jA'$,

an similarly $F_n(\Lambda \otimes_K \Lambda') = \bigoplus_{i+j=n} F_i\Lambda \otimes_{O_v} F_j\Lambda'$. Since $K \otimes_{O_v} F_n\Lambda = F_nA$ and $K \otimes_{O_v} F_n\Lambda' = F_nA'$ for all n, we get that $F_i\Lambda \otimes_{O_v} F_j\Lambda'$ is an O_v-submodule of $F_iA \otimes_K F_jA'$ for all i, j, and moreover the filtration defined on $\Lambda \otimes_{O_v} \Lambda'$ is induced by the filtration defined on $A \otimes_K A'$. The unramifiedness of $\Lambda \otimes_{O_v} \Lambda'$ is easily seen. □

We get now the following well-known result (see [6, Proposition 2.1])

Corollary 1.10. *If A is a finitely dimensional K-algebra, A' a K-central simple subalgebra of A, and Λ' an unramified reduction of A', then there exists Λ an unramified reduction of A such that $\Lambda' = \Lambda \cap A'$.*

Proof. It is a classical result that $A = A' \otimes_k A''$, where A'' is the centralizer of A' in A, and now we apply Proposition 1.9. □

The property of an unramified F-reductor of being a valuation ring is completely described by its reduction over O_v.

Proposition 1.11. *Let Δ be a skewfield that contains K in its center, $F\Delta$ a finite filtration on Δ, and $\Lambda \subset \Delta$ an unramified F-reductor. Then Λ is a valuation ring (for Δ) if and only if $\overline{\Lambda}$ is a skewfield.*

Proof. Assume that Λ is a valuation ring for Δ with maximal ideal m. Obviously $m_v\Lambda \subset m$, and we aim to show that the foregoing inclusion is an equality. Pick an element $x \in m$. Then $x^{-1} \in \Delta - \Lambda$, and thus we get an $n \in \mathbb{N}$ such that $x \in F_n\Lambda$ and $x^{-1} \in F_nA - F_n\Lambda$. Write $x = \sum_{i=1}^r a_ix_i$, $a_i \in O_v$, and $x^{-1} = \sum_{i=1}^r b_ix_i$, $b_i \in K$ (not all in O_v!), where $\{x_1, \dots, x_r\}$ is an O_v-basis for $F_n\Lambda$, and a K-basis for F_nA. By standard arguments we get an $j \in \{1, \dots, r\}$ such that $b_i^* = b_j^{-1}b_i \in O_v$ for all i and $b_j^{-1} \in m_v$. Then $b_j^{-1} = (\sum_{i=1}^r a_ix_i)(\sum_{i=1}^r b_i^*x_i)$. By taking the residues modulo $m_vF_n\Lambda$ we get that $0 = (\sum_{i=1}^r \widehat{a}_i\overline{x}_i)(\sum_{i=1}^r \widehat{b}_i^*\overline{x}_i)$. As $b_j^* = 1$, we must have that $\sum_{i=1}^r \widehat{a}_i\overline{x}_i = 0$, and this implies that $a_i \in m_v$ for all i.

Conversely, suppose that $\overline{\Lambda}$ is a skewfield. Since the valuation filtration $F^v\Delta$ is Γ-separated, $F_0^v\Delta = \Lambda$, and $G_v(\Delta)_0 = \overline{\Lambda}$ is a domain, we can apply Corollary 1.4 from [1] and get that Λ is a valuation ring of Δ. □

Let A be an affine K-algebra generated by $a_1, \dots, a_n$, $K < \underline{X} >$ the free K-algebra on the set $\underline{X} = \{X_1, \dots, X_n\}$ and $\pi : K < \underline{X} > \longrightarrow A$ the canonical K-algebras morphism given by $\pi(X_i) = a_i$, $i = 1, \dots, n$. Restriction of π to $O_v < \underline{X} >$ defines a subring Λ of A, i.e. $\Lambda = \pi(O_v < \underline{X} >)$. As before, the subring Λ yields a valuation filtration F^vA on A given by $F_\gamma^vA = (f_\gamma^vK)\Lambda$, $\gamma \in \Gamma$. It is easy to see that for a graded K-algebra A that has a finite PBW-basis, the subring Λ is an unramified F-reductor with respect to the grading filtration FA.

Note now that via Proposition 1.5 we get a new proof of the following result (Theorem 3.4.7 from [7])

Theorem 1.12. *For a graded K-algebra A that has a finite PBW-basis, the valuation filtration F^vA is Γ-separated and strong.*

By $G_v(A)$ we denote the associated graded ring determined by the valuation filtration F^vA. The next two results can be proved similarly to their correspondents

(Theorem 2.2 and Proposition 2.4) from [1], but we record them here in order to emphasize that now we do not need extra-conditions on the filtered parts of A.

Proposition 1.13. *Let A be a K-algebra with a finite filtration and $\Lambda \subset A$ an unramified F-reductor of A. If $\overline{A} = \Lambda/m_v\Lambda$ is a domain and A is an Ore domain, then every valuation v on K extend to $Q = Q_{cl}(A)$, the classical ring of fractions of A.*

Proposition 1.14. *If A is a K-algebra with a finite filtration and $\Lambda \subset A$ an unramified F-reductor of A, then the associated graded ring $G_v(A)$ is isomorphic to the twisted group ring $\overline{A} * \Gamma$, where $\overline{A} = \Lambda/m_v\Lambda$.*

It is a common strategy to deduce properties of A from properties of $G_F(A)$ whenever possible. Let us focus on the graded situation now.

Let R be an $\mathbb{N}$-graded K-algebra with $K \subseteq R_0$. Suppose that the gradation is *finite*, i.e. $\dim_K R_n < \infty$ for all $n \in \mathbb{N}$ and let $\Lambda \subset R$ be a graded subring. Then Λ is called a *graded reductor* if $\Lambda \cap K = O_v$ and $\Lambda \cap R_n$ is an O_v-lattice in R_n for all $n \in \mathbb{N}$. The ring $\overline{R} = \Lambda/m_v\Lambda$ is called the (graded) *reduction* of R with respect to Λ. The valuation filtration F^vR on R is similarly defined by $F^v_\gamma R = (f^v_\gamma K)\Lambda$, $\gamma \in \Gamma$.

Definition 1.15. *Let R be an $\mathbb{N}$-graded K-algebra with $K \subseteq R_0$, and* $\dim_K R_n < \infty$ *for all $n \in \mathbb{N}$. If $\Lambda \subset R$ is a graded reductor, then we say that Λ is an* unramified graded reductor *if $\Lambda \cap R_n$ is an unramified reduction of R_n for all $n \in \mathbb{N}$.*

It is rather easy to see that we have similar properties for unramified graded reductors to the ones already proved for unramified F-reductors. We mention here only one of them, but the interested reader is invited to do it on his own.

Proposition 1.16. *Let R be an $\mathbb{N}$-graded K-algebra with a finite gradation and $\Lambda \subset R$ an unramified graded reductor. Then the valuation filtration F^vR on R is Γ-separated.*

Proof. $\Lambda \cap R_n$ is a free O_v-module for all $n \in \mathbb{N}$. Lemma 1.3 entails that the valuation filtration F^vR_n is Γ-separated for all $n \in \mathbb{N}$, and by using [1, Proposition 3.7(ii)] we have that F^vR is Γ-separated. □

We also remark that for $\Gamma = \mathbb{Z}$ a graded reductor necessarily defines a separated filtration. We mention another interesting case when a graded reductor defines a Γ-separated filtration, the case of connected positively graded algebras. Recall that a K-algebra R is called a *connected* positively graded algebra if $R = K \oplus R_1 \oplus \cdots \oplus R_n \oplus \cdots = K[R_1]$ and $\dim_K R_1 < \infty$. Let us assume that $\dim R_1 = n$, $\underline{X} = \{X_1, \dots, X_n\}$ are indeterminates over K, and take $\pi : K < \underline{X} > \longrightarrow R$ a presentation of R. If we set $\Lambda = \pi(O_v < \underline{X} >)$, then $\dim_{\overline{K}} \overline{R}_1 = n$ since no elements of degree one in the gradation of $K < \underline{X} >$ are in $\mathcal{R}$, the ideal of relations of R. Nevertheless, $\dim_K R_n$ and $\dim_{\overline{K}} \overline{R}_n$ may be different for $n > 1$. However, we can prove the following.

Proposition 1.17. *Let R be connected positively graded K-algebra and $\Lambda \subset R$ defined as before. Then F^vR is Γ-separated.*

Proof. Note first that Λ is a graded ring, where the gradation is the one inherited from $O_v < \underline{X} >$ via π. All we have to do is to show that Λ is a graded subring

of R, i.e. $\Lambda_n = \Lambda \cap R_n$ for all n. For $n = 0$ it means that $O_v = \Lambda \cap K$ which is obviously true. For $n > 0$, pick an element $x \in \Lambda \cap R_n$. Since $x \in R_n$ there exists an element $a \in O_v$ such that $ax \in \Lambda_n$, and since $x \in \Lambda$ there exists $y \in O_v < \underline{X} >$ such that $x = \pi(y)$. Writing y as a sum of homogeneous components, $y = \sum_{i \geq 0} y_i$, and multiplying the relation by $a \in O_v$ we get that $a\pi(y_i) = 0$ for all $i \neq n$, therefore $\pi(y_i) = 0$ for all $i \neq n$, that is $x = \pi(y_n) \in \Lambda_n$. Thus we get that Λ is a graded reductor and Λ_n is O_v-free for all n (note that Λ_n is a finitely generated O_v-module), and consequently $F^v R$ is Γ-separated. □

Note that graded reductor Λ defined above is not necessarily unramified, although all its graded components are free O_v-modules.

For the sake of completeness we recall here the following result (Lemma 3.3 from [1])

Lemma 1.18. *Let V be a finite dimensional K-vector space, $V' \subset V$ a K-vector subspace and $M \subset V$ an O_v-submodule.*

- *(i) If M is an O_v-lattice in V, then the quotient module $M/M \cap V'$ is an O_v-lattice in V/V'.*
- *(ii) If $M/M \cap V'$ is an O_v-lattice in V/V' and $M \cap V'$ is an O_v-lattice in V', then M is an O_v-lattice in V.*

Proposition 1.19. *Let A be a K-algebra with a finite filtration FA, and $\Lambda \subset A$ a subring.*

- *(i) If $\Lambda \subset A$ is an unramified F-reductor, then $G_F(\Lambda) \subset G_F(A)$ and $\widetilde{\Lambda} \subset \widetilde{A}$ are unramified graded reductors.*
- *(ii) If $G_F(\Lambda) \subset G_F(A)$ or $\widetilde{\Lambda} \subset \widetilde{A}$ are unramified graded reductors, then $\Lambda \subset A$ is an unramified F-reductor.*

Proof. (i) In order to show that $G_F(\Lambda) \subset G_F(A)$ is an unramified graded reductor we use Lemma 1.18(i). That $\widetilde{\Lambda} \subset \widetilde{A}$ is an unramified graded reductor follows by the definition of Rees algebra.
(ii) By induction using Lemma 1.18(ii). □

Consequently any unramified F-reductor give rise to an unramified graded reductor. On the other side, an unramified graded reductor is an unramified F-reductor, where FR is the grading filtration.

As applications of the above Proposition 1.19 we mention here two results: the first one is Theorem 2.6 from [5], and the second one is Proposition 3.2 from [3]. It is worthwhile to remark that in both cases our results hold for every Γ-valuation, while their results are given only for discrete valuations.

Proposition 1.20. *Let A be a K-algebra with a finite filtration FA, and $\Lambda \subset A$ a subring such that $\Lambda \cap K = O_v$ and $K\Lambda = A$.*

- *(i) If $G_F(\Lambda)_n$ is a finitely generated O_v-module for all n, then the valuation filtration $F^v A$ is Γ-separated.*
- *(ii) If $F_n\Lambda$ are finitely generated O_v-modules for all n, then the filtrations $F^v A$ and $f^v G_F(A)$ are Γ-separated.*

Proof. (i) Since $G_F(\Lambda)_n$ are all finitely generated O_v-modules, we easily get that $F_n\Lambda$ are all finitely generated O_v-modules, and thus Λ is an F-reductor. Once again we apply Proposition 1.1(i) and deduce that $F_n\Lambda$ are O_v-free, and this is enough in order to conclude that the valuation filtration F^vA is Γ-separated.
(ii) From (i) we know that F^vA is Γ-separated. Moreover, $G_F(\Lambda) \subset G_F(A)$ is a graded reductor with the graded components free O_v-modules, therefore the valuation filtration $f^vG_F(A)$ is Γ-separated. □

Proposition 1.21. *Let $A = K[a_1, \ldots, a_r]$ be an affine K-algebra, and Λ as before but with the generator filtration, i.e. $F_{-1}\Lambda = 0$, $F_0\Lambda = O_v$, and $F_n\Lambda$ is the O_v-module generated by the elements $a_{i_1} \cdots a_{i_t}$ with $1 \leq t \leq n$. If the associated graded ring $G_F(\Lambda)$ is a flat O_v-module, then the valuation filtration F^vA is Γ-separated.*

Proof. We consider on A the generator filtration FA. In principle, we do not know that $G_F(\Lambda)$ is contained in $G_F(A)$, and that is why we are asking for flatness. To enter the details, take $x \in F_n\Lambda \cap F_{n-1}A$. We want to show that $x \in F_{n-1}\Lambda$. Suppose that this is not true. Then there exists an $a \in O_v - \{0\}$ such that $ax \in F_{n-1}\Lambda$ which means that $ax = 0$ in the associated graded ring $G_F(\Lambda)$. As we know that $G_F(\Lambda)$ is a flat O_v-module, i.e. torsion-free, we get a contradiction. Therefore $F_n\Lambda \cap F_{n-1}A = F_{n-1}\Lambda$, that is $G_F(\Lambda) \subset G_F(A)$ is a graded reductor with finitely generated (hence free) graded components, and this is enough to see that $\Lambda \subset A$ is an F-reductor with all filtered parts O_v-free. □

Here there are some (old) examples of algebras that admit unramified reductors.

Examples 1.22. (i) Consider a field k with $\operatorname{Char} k \neq 2$, and set $K = k(X)$ the field of rational functions over k. Let $L = K(\xi)$, where ξ is a root of the irreducible polynomial $T^2 + (X-1)T + X \in K[T]$. Now consider the valuation ring of K given by $O_v = k[X]_{(X)}$, and $\Lambda = O_v[\xi]$. It is easy to see that $O_v \subset \Lambda$ is an integral extension, $\overline{K} = k_v \simeq k$, $\overline{\Lambda} = \Lambda/m_v\Lambda \simeq k[u]$, where u is an idempotent, and therefore Λ is an unramified reductor of L.
(ii) Consider g a finite dimensional *Lie algebra* over a field K and $A = U(g)$ the enveloping algebra of g. Let O_v be a valuation ring of K. We define a finite dimensional Lie algebra g_{O_v} over O_v with the same basis and the induced bracket. Let us fix a K-basis $\{x_1, \ldots, x_n\}$ for g. We have structure constants $\lambda_{ij}^k \in K$ with $[x_i, x_j] = \sum_{k=1}^n \lambda_{ij}^k x_k$. Without loss of generality we may assume that $\lambda_{ij}^k \in O_v$ (up to multiplying all x_i by a suitable constant in O_v) but not all in m_v. Set $g_{O_v} = O_vx_1 + \cdots + O_vx_n$. This is a Lie O_v-algebra with the induced bracket. Furthermore, g_{O_v} is an O_v-lattice in g. On $\overline{g} = g_{O_v}/m_vg_{O_v}$ we define a Lie algebra structure over $\overline{K}$ by setting $[\overline{x}_i, \overline{x}_j] = \sum_{k=1}^n \widehat{\lambda_{ij}^k}\overline{x}_k$, where the $\overline{x}_i$ are the images of the x_i in $\overline{g}$ and $\widehat{\lambda_{ij}^k}$ are the images of λ_{ij}^k in $\overline{K}$. By our assumptions $\overline{g}$ is not the trivial Lie algebra. Of course, $\overline{g}$ depends on the choice of the K-basis in g.

Let $\Lambda = U_{O_v}(g_{O_v})$ be the enveloping algebra of g_{O_v}. Consider on A the standard filtration FA and on Λ the induced filtration $F\Lambda$. We have that the filtration FA is finite, $G_F(\Lambda) = O_v[X_1, \ldots, X_n]$ is a subring of the polynomial ring $G_F(A) = K[X_1, \ldots, X_n]$ and obviously it is an unramified graded reductor. From Proposition 1.19(ii) we get that Λ is an unramified F-reductor. Furthermore, the filtration F^vA

is Γ-separated, $G_v(A)$ is a domain isomorphic to $U_{k_v}(\overline{g})$ and thus we can extend v to $D(g) = Q_{cl}(U(g))$.
(iii) For the *Weyl algebra* $R = A_n(K)$ we take $\Lambda = A_n(O_v)$. We claim that Λ is an unramified graded reductor of R. First note that Λ is a free O_v-module, therefore it defines a good reduction of R. On the other side, we have that $\mathrm{rank}_{O_v} \Lambda_n = \dim_K R_n$ for all $n \in \mathbb{N}$, since Λ and R have the same PBW-basis, and so Λ is an unramified graded reductor of R. It is also an F-reductor since the associated graded ring of $A_n(K)$ with respect to the Bernstein filtration is a polynomial ring.
(iv) Let K and O_v as before. Set $R = K < X, Y > /(XY - qYX)$ for the *quantum plane*, where q is a unit in O_v. Then the subring $\Lambda = O_v < X, Y > /(XY - qYX)$ is an unramified F-reductor with respect to the generator (grading) filtration, and an unramified graded reductor with respect to the mixed gradation.
(v) Let K, O_v and q as before, and set $A = K < X, Y > /(XY - qYX - 1)$ for the *quantum Weyl algebra*. Then $\Lambda = O_v < X, Y > /(XY - qYX - 1)$ is an unramified F-reductor with respect to the generator filtration.

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